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Algebraic integer valued holomorphic functions of exponential type

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1. Introduction

In this section we give a brief history of our subject.
In 1915, G.Polya proved following theorem.

Theorem 1. ([9]) Suppose that entire function f satisfies the following conditions:

(1) f is an entire function of exponential type A,

(2) $f(\mathbb{N}) \subset \mathbb{Z}$

If A is less than log 2, then f is a polynomial with rational integer coefficients.

In 1946, C. Pisot generalized Polya's theorem using Laplace trans form and transfinite diameter. ([8])
In 1974, V. Avanissian and R. Gay generalized Pisot's result to entire functions of exponential type in $\mathbb{C}^m$ using the theory of analytic functionals. ([2])
In 1977, F.Gramain obtained several results about entire function f of exponential type in $\mathbb{C}^1$ satisfying $f(\mathbb{N}) \subset O_K$.
Here O_K denotes the ring of algebraic integers contained in algebraic number field K over $\mathbb{Q}$. ([4])
In 1988, A.Bazylewicz generalized F.Gramain's result to entire functions of exponential type in $\mathbb{C}^m$. ([3])
We will generalize Bazylewicz's result to non entire functions of

exponential type in $\mathbb{C}^m$ using the theory of analytic functionals with non-compact carrier(a kind of Fourier ultra hyperfunction). To close this section, we notice the special relation between the theory of arithmetic holomorphic function and the Ising model in statistical physics. ([7]).

2. Notations

Let K be a number field over $\mathbb{Q}$ with degree $[K:\mathbb{Q}] = d = r+2s$. $K^{(i)}$ $(1 \le i \le r)$ and $\overline{K^{(r+j)}} = K^{(r+s+j)}$ $(1 \le j \le s)$ are its conjugate field.

δ is defined as follows:

$$\delta = \begin{cases} d & \text{if } K \subset \mathbb{R} \\ d/2 & \text{if } K \not\subset \mathbb{R} \end{cases}$$

For algebraic integer a, we put

$$\overline{|a|} = \max_{1 \le i \le s} |a_i| \quad (a_i\text{'s are conjugates over } \mathbb{Q})$$

Then following inequality valids :

$$\log |a| \ge -(\delta-1)\log\overline{|a|}$$

(inegalite de la taille)

O_A denotes the ring of algebraic integers.

$O_K^{(j)}$ denotes the ring of algebraic integers in $K^{(j)}$.

For $p(z) = \Sigma\ a_n z^n \in K[z]$, we put $p^{(j)}(z) = \Sigma\ a_n^{(j)} z^n \in K^{(j)}[z]$, where $a_n^{(j)}$ is conjugate of a_n.

$\tau(A)$ denotes the transfinite diameter of compact set A.

For the details of transfinite diameter we refer the reader to [1].

Supporting function $H_A(z)$ of set A is defined by

$$H_A(z) = \sup_{\zeta \in A} \mathrm{Re} \langle z, \zeta \rangle$$

$$|z| = |z_1| + |z_2| + \cdots + |z_m|, \quad (z_1, z_2, \ldots z_m) \in \mathbb{C}^m$$

3. Some results on O_K valued entire function of exponential type

In 1976-1977, F. Gramain obtained following theorems.

Theorem 2. ([4]) Let K_1 be a compact convex set in $\mathbb{C}$ contained in $\{\zeta \in \mathbb{C}: |\mathrm{Im}\ \zeta| < \pi\}$. Suppose that f(z) satisfies following conditions (i),(ii),(iii).

(i) f(z) is entire function of exponential type H_{K_1}.

(ii) $f(\mathbb{N}) \subset O_K$

(iii) $\limsup_{n \to \infty} 1/n \log \overline{|f(n)|} \leq c$

If $\log \tau(\exp(K_1)) < -c(\delta-1)$, then $f(z) = \Sigma P_b(z)\, b^z$.

Where $P_b(z) \in K(\vartheta)[z]$ and $\vartheta = \{a$: a algebraic integer, $|a| \leq e^c$, a is contained in $\exp(K_1)$ together with its conjugates over K.$\}$.

Theorem 3. (A generalization of Polya 's theorem [5])

(i) f(z) is entire function of exponential type α

(ii) $f(\mathbb{N}^m) \subset O_K$

(iii) $\limsup_{n \to \infty} 1/ n \log \overline{|f(n)|} \leq c$

If $\log (e^\alpha - 1) < -(\delta-1)\log(1+e^c)$, then $f(z) \in K[z]$.

In 1988, Bazylewicz generalized theorem 2 to entire functions of exponential type in $\mathbb{C}^n$. Namely, he obtained following theorem 4.

Theorem 4. ([3]) Suppose that $f(z)$ is entire function of exponential type satisfying following conditions:

(i) For any $\varepsilon > 0$, there exists $C_\varepsilon \geq 0$ such that

$$|f(z)| \leq C_\varepsilon \exp\left(\sum_{i=1}^{m} H_{T_i}(z_i) + \varepsilon |z|\right), \quad z=(z_1, z_2, \ldots z_m) \in \mathbb{C}^m$$

where T_i's are compact convex sets in $\mathbb{C}$.

(ii) $f(\mathbb{N}^m) \subset O_K$

(iii) there exists a positive number c such that

$$\limsup_{n \to \infty} 1/n \log \overline{|f(n)|} \leq c$$

If $\log \tau(\exp(T_i)) < -c(\delta-1)$ $(1 \leq i \leq m)$, then

$f(z) = \Sigma P_b(z) b^z$,

where $P_b(z) \in K(\vartheta)[z]$ and ϑ is a finite set of O_A^m.

4. Main result

Using the theory of analytic functional with non-compact carrier we can generalize theorem 4 to holomorphic functions of exponential type defined in the product of half plane. Namely, we can generalize theorem 4 as follows.

Theorem 5. (Main result) Suppose that $0 \leq k' < 1$ and function $f(z)$ satisfies following conditions :

(i) $f(z)$ is holomorphic in $\Pi_{i=1}^{m}\{z_i : \operatorname{Re} z_i < -k'\}$

(ii) For any $\varepsilon > 0$ and $\varepsilon' > 0$, there exists $C_{\varepsilon,\varepsilon'} \geq 0$ such that

$|f(z)| \leq C_{\varepsilon,\varepsilon'} \exp(H_L(z)+\varepsilon|z|) \quad (\mathrm{Re}\, z_i \leq -k' - \varepsilon')$

where L is a convex set contained in $\Pi_{i=1}^{m}\{\zeta_i \in \mathbb{C} : |\mathrm{Im}\, \zeta_i| \leq b_i < \pi,$ $\mathrm{Re}\, \zeta_i \geq a_i\}$.

(iii) $f((-\mathbb{N})^m) \subset O_K$

(iv) $\limsup_{n\to\infty} 1/n \log \overline{|f(-n)|} \leq c$

If $\tau(\exp(-T_i)) < -c(\delta-1)$ $(1 \leq i \leq m)$, then $f(z) = \sum P_b(z)\, b^z$ where $P_b \in K(\vartheta)[z]$, and ϑ is a finite set of O_A^m.

5. Transformations of analytic functionals with non-compact carriers

In this section we recall Avanissian-Gay transform and Laplace transform of analytic functionals with non-compact carrier.

Let L be a convex closed set in $\mathbb{C}^m$ bounded in imaginary direction. we define holomorphic test function space $Q(L;k')$ as follows:

$$Q(L;k') = \varinjlim_{\varepsilon>0,\varepsilon'>0} Q_b(L_\varepsilon;\, k'+\varepsilon')$$

$$Q_b(L_\varepsilon;k'+\varepsilon') = \{f \in \mathcal{O}(L_\varepsilon) \cap C(\overline{L_\varepsilon}) : \sup_{\zeta\in L_\varepsilon} |f(\zeta)\exp((k'+\varepsilon')\zeta)| \leq \infty\}$$

$Q'(L:k)$ denotes the dual space of $Q(L:k')$. The element of dual space is called analytic functional with carrier L. Now we define Laplace transform $\tilde{T}(z)$ of $T \in Q'(L:k')$ as follows:

$$\tilde{T}(z) = \langle T_\zeta, \exp(z\zeta)\rangle$$

where $z\zeta = z_1\zeta_1+\ldots+z_m\zeta_m$

Following Paley-Wiener type theorem characterizes Laplace transform of $Q'(L:k')$.

Theorem 6. ([10])

Let T belong to $Q'(L:k')$, then $\tilde{T}(z)$ is holomorphic function in $D = \Pi_{i=1}^{m}\{z_i \in \mathbb{C} : \mathrm{Re}\, z_i < -k'\}$ and satisfies following estimate:

for any $\varepsilon > 0$, $\varepsilon' > 0$, there exists $C_{\varepsilon,\varepsilon'} \geq 0$ such that

$$(*) \qquad |\tilde{T}(z)| \leq C_{\varepsilon,\varepsilon} \exp(H_L(z) + \varepsilon\, |z|) \qquad (\mathrm{Re}\, z_i \leq -k' - \varepsilon')$$

Conversely, if holomorphic function f(z) in D satisfies (*), then f(z) is a Laplace transform of some $T \in Q'(L:k')$.

For the details of Laplace transform of analytic functionals, we refer the reader to [10]

To define Avanissian-Gay transform we put following assumptions :

(i) $0 \leq k' < 1$

(ii) $pr_i(L) \subset \{\zeta_i \in \mathbb{C} : |\mathrm{Im}\ \zeta_i| \leq b_i < \pi,\ \mathrm{Re}\ \zeta_i \geq a_i\}$

where $pr_i(L)$ denotes the i-th projection of L.

Avannisian-Gay transform $G_T(w)$ of $T \in Q(L:k)$ is defined as follows :

$$G_T(w) = \langle T_\zeta,\ \Pi_{i=1}^{m}(1 - w_i\zeta_i)^{-1} \rangle$$

Avannisian-Gay transform has following properties:

Proposition 1.([7],[10])

(i) $G_T(w)$ is holomorphic in $\Pi_{i=1}^{m}(\mathbb{C}/\exp(-L_i))$.

(ii) $G_T(w) = (-1)^m \Sigma \quad T(-n_1, -n_2, , , -n_m) w_1^{-n_1} w_2^{-n_2} \ldots w_m^{-n_m}$

(iii) for any $\varepsilon > 0$, and $\varepsilon' > 0$, there exists $C_{\varepsilon,\varepsilon'} \geq 0$ such that

$$|G_T(w)| \le C_{\varepsilon,\varepsilon'} |w_1|^{-k'-\varepsilon'} |w_2|^{-k'-\varepsilon'} \cdots |w_m|^{-k'-\varepsilon'}$$

(iv) (inversion formula)

$$T(z) = (2\pi i)^{-m} \int_{\Gamma} G_T(w_1, w_2, ,, w_m) w_1^{-z_1-1} w_2^{-z_2-1} \cdots w_m^{-z_m-1}\, dw_1 dw_2 \cdots dw_m$$

where $\Gamma = \Gamma_1 \times \Gamma_2 \times \ \times \Gamma_m$, $\Gamma_i (1 \le i \le m)$ is the contour surrounding $\exp(-L_i)$.

6.Proof of main result.

In this section we will prove theorem 5. Following proposition is essential in our discussion.

Proposition 2(Bazylewicz[3])

$S_{i,j} (1 \le i \le m)$ are compact sets in $\mathbb{C}$ and $\tau_{i,j} = \tau(S_{i,j})$ are their transfinite diameters. Suppose that $S_{i,j}$ satisfies following conditions:

(i) $S_{i,j} = \overline{S_{i,j}}$ $(i \le j \le r)$ (i.e. $S_{i,j} \subset \mathbb{R}$)

$S_{i,j+r} = \overline{S_{i,r+j+s}}$ $(1 \le j \le s)$

(ii) $\prod_{j=1}^{d} \tau_{i,j} < 1$ $(1 \le i \le m)$

We suppose that

$g^{(j)}(z) = \sum_{n \in \mathbb{N}^m} a_n^{(j)} z^{-1-n}$ is holomorphic in $\prod_{i=1}^{m} (\mathbb{C} / S_{i,j})$

where $a_n^{(j)}$'s are algebraic integers in K. Then there exists polynomials $P(z_1, z_2, ,, z_m)$, $Q_i(z_i)$ satisfying following conditions :

(1) $Q_i(z_i)$are monic (coefficient of highest degree term is unit.)

(2) $\deg_{z_i} P(z_1, z_2, ,, z_m) \leq \deg Q_i(z_i)$

(3) $g^{(j)}(z) = P^{(j)}(z_1, z_2, ,, z_m)/\prod_{i=1}^{m} Q_{i,}^{(j)}(z_i)$

Now we give the proof of theorem 5. By Theorem 4, there exists $T \in Q\ (L: k\ )$ such that $f(z) = T(z)$. We put $g(w) = G_T(w)$. By (ii) in proposition 1, we have

$g(w) = (-1)^m \sum_{n \in \mathbb{N}^m} f(-n) w^{-1-n}$.

Now we put

$g^{(j)}(w) = (-1)^m \sum_{n \in \mathbb{N}^m} f(-n)^{(j)}\ w^{-1-n}$ $\qquad (1 \leq j \leq d)$.

Each $g^{(j)}$ is holomorphic in $\prod_{i=1}^{m}\{|w_i| \geq e^c\}$.

Since g(w)is Avanissian-Gay transform of T, g(w) and $g^{(j)}$ (w) is holomorphic in

$\prod_{i=1}^{m}(\mathbb{C}/\overline{\exp(-L_i)})$, $\overline{\prod_{i=1}^{m}(\mathbb{C}/\overline{\exp(-L_i)})}$.

Here $\overline{\exp(-L_i)}$ is a closure of $\exp(-L_i)$, and $\overline{\mathbb{C}/\overline{\exp(-L_i)}}$ is complex conjugate of $\overline{\mathbb{C}/\ \exp(-L_i)}$.

We put $S_i = \overline{\exp(-L_i)}$ and

$$S_i^{(j)} = \begin{cases} S_i & \text{if } K = K^{(j)} \\ \overline{S_i} & \text{if } \overline{K} = K^{(j)} \\ \\ |w_i| \leq e^c, & \text{other case.} \end{cases}$$

Then $g^{(j)}$ and $S_i^{(j)}$ satisfy all assumptions in proposition 2. So we have

$G_T(w) = g(w) = P(w_1, w_2, \ldots w_m)/\ \prod_{i=1}^{m} Q_i(w_i)$

where P and Q_i's are polynomials satisfying the conditions in proposition 2.

By inversion formula in proposition 1 and residue theorem, we obtain

$$f(z) = (2\pi i)^{-1} \int G_T(w) w_1^{-z_1-1} w_2^{-z_2-1} \dots w_m^{-z_m-1} dw_1 dw_2 \dots dw_m$$

$$= (2\pi i)^{-1} \int P(w_1, \dots w_m)/\prod_{i=1}^{m} Q_i(w_i)\ w_1^{-z_1-1} \dots w_m^{-z_m-1} dw_1 \dots dw_m$$

$$= \sum_{\gamma:\ \text{algebraic integer}} P_\gamma(z)\, \gamma^z.$$

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